# Systems of ODEs

1. Find eigenvalues → • distinct and real
   • Complex conjugate pairs
   • repeated and full set of eigenvectors
   • repeated and geo mult. < alg. mult. → generalized eigenvector
2. Determine eigenvectors
3. General solution
4. Use ICs.

Homogeneous eqn: $\vec{x}' = P(t)\vec{x}$

Nonhomogeneous eqn: $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$

- Diagonalization
- Variation of parameters
- Method of undetermined coefficients
- Laplace transform.

Ex Use variation of parameters to solve

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$

Show that you have a fundamental set of solutions.

First, solve the homogeneous problem.

The eigenvalues are

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda+1)(\lambda-3) = 0$$

$$\lambda_2 = -1, \quad \lambda_1 = 3.$$

and the eigenvectors are

$\underline{\lambda_1 = 3}$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \vec{v_1} = 0 \Rightarrow \vec{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\underline{\lambda_2 = -1}$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \vec{v_2} = 0 \Rightarrow \vec{v_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Therefore,

$$\Psi(t) = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix}$$

Now, we suppose

$$\vec{x} = \Psi(t)\vec{u}(t).$$

$$\vec{x}' = \Psi'\vec{u} + \Psi\vec{u}'$$

$$\vec{x}\,' = \begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}\vec{x} + \begin{pmatrix}2\\ -1\end{pmatrix}e^{t}$$

$$\Psi'\vec{u} + \Psi\vec{u}\,' = \begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}\Psi\vec{u} + \begin{pmatrix}2\\ -1\end{pmatrix}e^{t}$$

$$\begin{pmatrix}3e^{3t} & e^{-t}\\ 6e^{3t} & -2e^{-t}\end{pmatrix}\vec{u} + \begin{pmatrix}e^{3t} & -e^{-t}\\ 2e^{3t} & 2e^{-t}\end{pmatrix}\vec{u}\,'$$

$$= \begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}\begin{pmatrix}e^{3t} & -e^{-t}\\ 2e^{3t} & 2e^{-t}\end{pmatrix}\vec{u} + \begin{pmatrix}2\\ -1\end{pmatrix}e^{t}$$

$$= \begin{pmatrix}3e^{3t} & e^{-t}\\ 6e^{3t} & -2e^{-t}\end{pmatrix}\vec{u} + \begin{pmatrix}2\\ -1\end{pmatrix}e^{t}$$

$$\begin{pmatrix}e^{3t} & -e^{-t}\\ 2e^{3t} & 2e^{-t}\end{pmatrix}\vec{u}\,' = \begin{pmatrix}2\\ -1\end{pmatrix}e^{t}$$

$$\vec{u}' = \frac{1}{2e^{2t}+2e^{2t}} \begin{pmatrix} 2e^{-t} & e^{-t} \\ -2e^{3t} & e^{3t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$$

$$= \frac{1}{4} e^{-2t} e^{t} \begin{pmatrix} 3e^{-t} \\ -5e^{3t} \end{pmatrix}$$

$$= \frac{1}{4} e^{-t} \begin{pmatrix} 3e^{-t} \\ -5e^{3t} \end{pmatrix}$$

$$u_1' = \frac{3}{4} e^{-2t} \qquad u_2' = \frac{-5}{4} e^{2t}$$

$$u_1 = \frac{-3}{8} e^{-2t} + c_1 \qquad u_2 = \frac{-5}{8} e^{2t} + c_2$$

$$\vec{x} = \Psi(t)\vec{u}$$

$$= \begin{pmatrix} 1e^{3t} & -1e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} -\frac{3}{8}e^{-2t} + c_1 \\ -\frac{5}{8}e^{2t} + c_2 \end{pmatrix}$$

$$= \Phi(t)\left(\vec{v} + \vec{c}\right)$$

$$= \underbrace{\Phi(t)\,\vec{c}}_{\text{homogeneous soln}} + \underbrace{\Phi(t)\,\vec{v}}_{\text{particular soln}}$$

# Laplace Transform

**Def** $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$

$F(s) =$

Inverse Laplace transform : Rewrite $F(s)$ and use Laplace Transform Table

**Ex** Find the inverse Laplace transform of

$$F(s) = \frac{1-2s}{s^2+4s+5}$$

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$$F(s) = \frac{1-2s}{s^2+4s+4+1}$$

$a = -2$

$s-a = s+2$

$2s+4$

$s - -2$

$$= \frac{1-2s}{(s+2)^2+1}$$

$$= \frac{1-2s-4+4}{(s+2)^2+1}$$

$$= \frac{5-2(s+2)}{(s+2)^2+1}$$

$$= \frac{5}{(s+2)^2+1} - 2\frac{s+2}{(s+2)^2+1}$$

$$f(t) = 5\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}$$

$$= 5e^{-2t}\sin t - 2e^{-2t}\cos(t)$$

# Solving IVP using Laplace Transform

1. Transform the eqn.
2. Solve for $Y(s)$
3. Return to $t$-domain

Ex $y'' + w^2 y = \cos(2t)$, $w^2 \neq 4$
$y(0) = 1$
$y'(0) = 0$.

$$\mathcal{L}\{y'' + w^2 y\} = \mathcal{L}\{\cos(2t)\}$$

$$\mathcal{L}\{y''\} + w^2 \mathcal{L}\{y\} = \frac{s}{s^2+4}$$

$$s^2 Y(s) - s y(0) - y'(0) + w^2 Y(s) = \frac{s}{s^2+4}$$

$$s^2 Y(s) - s + w^2 Y(s) = \frac{s}{s^2+4}$$

$$Y(s)(s^2+\omega^2) = \frac{s}{s^2+4} + s$$

$$Y(s) = \frac{s}{(s^2+4)(s^2+\omega^2)} + \frac{s}{s^2+\omega^2}$$

$$\frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{As}{s^2+4} + \frac{Bs}{s^2+\omega^2}$$

$$= \frac{As^3 + A\omega^2 s + Bs^3 + 4Bs}{(s^2+4)(s^2+\omega^2)}$$

$$= \frac{(A+B)s^3 + (A\omega^2+4B)s}{(s^2+4)(s^2+\omega^2)}$$

$$A+B=0 \qquad A\omega^2+4B=1$$

$$A=-B \qquad -B\omega^2+4B=1$$

$$B = \frac{1}{4-\omega^2}$$

$$A = \frac{1}{\omega^2-4}$$

$$Y(s) = \frac{s}{(s^2+4)(s^2+w^2)} + \frac{s}{s^2+w^2}$$

$$= \frac{1}{w^2-4} \cdot \frac{s}{s^2+4} + \frac{1}{4-w^2} \cdot \frac{s}{s^2+w^2} + \frac{s}{s^2+w^2}$$

$$= \frac{1}{w^2-4} \cdot \frac{s}{s^2+4} + \left(\frac{1}{4-w^2} + 1\right) \frac{s}{s^2+w^2}$$

$$y(t) = \frac{1}{w^2-4} \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \left(\frac{1}{4-w^2}+1\right) \mathcal{L}^{-1}\left\{\frac{s}{s^2+w^2}\right\}$$

$$= \frac{1}{w^2-4} \cos(2t) + \left(\frac{1}{4-w^2}+1\right) \cos(wt)$$

## Picard Iterates

$$\frac{dy}{dt} = f(t,y) \quad , \quad y(0) = 0$$

$$\{\phi_n(t)\}$$

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$$

- usually assume $\phi_0(0) = 0$

$f(t,y) = t^2 y - t$

<u>Ex</u> $y' = t^2 y - t, \quad y(0) = 0 \quad f(s, \phi_n) = s^2\phi_n - s$

Determine $\phi_n(t)$ and show whether $\phi_n$ converges.

$$\phi_0(t) = 0$$

$$\phi_1(t) = \int_0^t -s\,ds = -\frac{t^2}{2}$$

$$\phi_2(t) = \int_0^t s^2\left(-\frac{s^2}{2}\right) - s\,ds$$

$$= -\frac{t^5}{2\cdot 5} - \frac{t^2}{2}$$

$$\phi_3(t) = \int_0^t s^2\left(\frac{-s^5}{2\cdot 5} - \frac{s^2}{2}\right) - s \, ds$$

$$= \frac{-t^8}{2\cdot 5\cdot 8} - \frac{t^5}{2\cdot 5} - \frac{t^2}{2}$$

$$\phi_n(t) = -\sum_{i=1}^{n} \frac{t^{2+3(i-1)}}{\prod_{j=1}^{i} 2+3(j-1)}$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{t^{2+3(n+1-1)}}{\prod_{j=1}^{n+1} 2+3(j-1)} \cdot \frac{\prod_{j=1}^{n} 2+3(j-1)}{t^{2+3(n-1)}}$$

$$= \frac{t^{2+3n}}{t^{3n-1}} \cdot \frac{1}{2+3(n+1-1)}$$

$$= t^3 \cdot \frac{1}{2+3n}$$

$\longrightarrow 0$ as $n \to \infty$

So $\Phi_n(t)$ converges by the ratio test.

## Modeling

Some quantity $a(t)$

$\rightarrow$ build model based on interpretation.

$$\frac{da}{dt} = \text{rate in} - \text{rate out}$$

Ex A tank initially contains 120 L of pure water.
A mixture of $\gamma$ g/L enters at a rate of 2 L/min and the

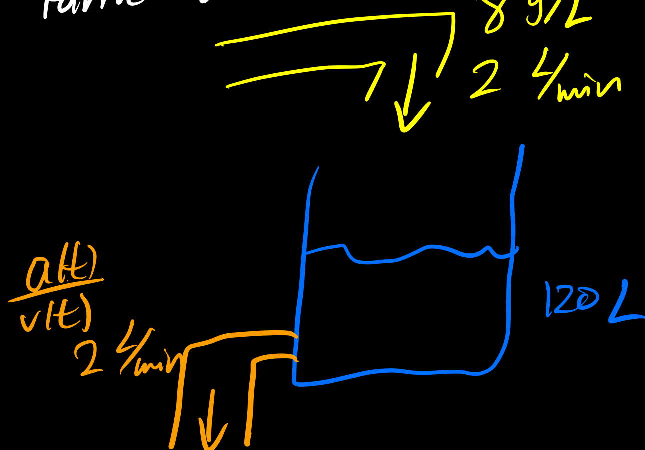
mixture in the tank exits at the same rate.

Find $a(t)$, the amount of salt in tank at time $t$.

$a$ – amount of salt (g)

$V$ – volume (L)

$V(0) = 120$




$$\gamma \frac{g}{\cancel{L}} \cdot 2 \frac{\cancel{L}}{min} = 2\gamma \frac{g}{min}$$ rate salt enters

$$\frac{a(t)}{V(t)} \frac{g}{L} \cdot 2 \frac{L}{min} = \frac{2a(t)}{V(t)} \frac{g}{min}$$ rate salt exits

$$\frac{dV}{dt} = 0 \implies V(t) = C \implies V(t) = 120$$

IVP

$$\frac{da}{dt} = 2\gamma - \frac{2a}{120} \quad , \quad a(0) = 0 \,.$$